Gauss's Law

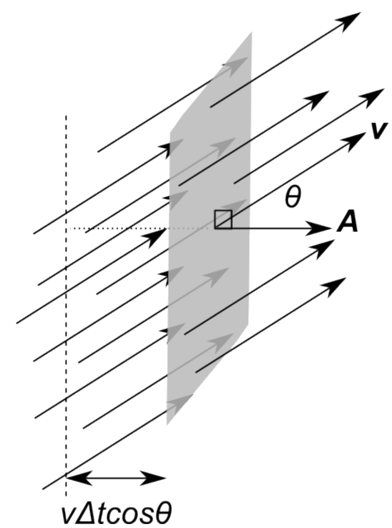
Flux

What field lines through a closed surface tell us about the contained charge

Flux and the concept of flow

We need to quantify the concept of field lines passing through a surface. The "quantity" of electric field passing through a surface should depend on 1) the magnitude of electric fild; 2) the area of the surface; and 3) the orientation of the surface relative to the direction of the electric field. The field lines passing through the surface will be maximized if they are perpendicular to the surface, because the full area of the surface will be exposed to the field lines. If the field lines are parallel to the surface, then no field lines will pass through.

We develop the quantity we call flux by analogy with fluid flow. We can calculate the volume flow rate of a fluid through a surface given the velocity of the fluid, the area of the surface, and the orientation of the surface relative to the velocity of the fluid. We define the flux as the volume flow rate of the fluid through the surface. See the following figure.



In this figure, we see that the volume of fluid passing through the surface in time *Δt* is given by:

$$\Delta V = v \cdot \Delta t \cdot \cos\theta \cdot A$$

and the flux is therefore:

$$flux = v \cdot A \cdot \cos\theta$$

If we define the area vector ***A*** as the vector normal to the surface with the magnitude equal to the area of the surface, then we can write:

$$flux = \vec{v} \cdot \vec{A}$$

We define the electric flux in the same way, even though nothing is actually "flowing".

$$\Phi_E = \vec{E} \cdot \vec{A}$$

Examples of flux calculations:

A circular disk with a radius of 2.5 cm is placed horizontally in a uniform electric field **E**=4.0N/C**i** + 3.0N/C**j**. Calculate the flux through the disk.

This definition works fine if we have a uniform electric field and a planar surface; however, if the surface is curved and the field is varying in space, then we need a different method. In this case, we split the surface into infinitesimal surface elements **dA**. The infinitesimal vector **dA** is normal to the surface element and the electric field has a unique value at the location of this surface element. The electric flux through this surface element is then:

$$d\Phi_E = \vec{E} \cdot \overrightarrow{dA}$$

To calculate the flux through the entire surface, we integrate:

$$\Phi_E = \iint \vec{E} \cdot \overrightarrow{dA}$$

The double integral sign reminds us that we must integrate over a two-dimensional surface, which will in principle require integrating over each of the two dimensions.

Example: A sphere is placed in a uniform field **E**=4.0N/*C***i** + 3.0N/C**j**, show that the flux through the sphere is zero.

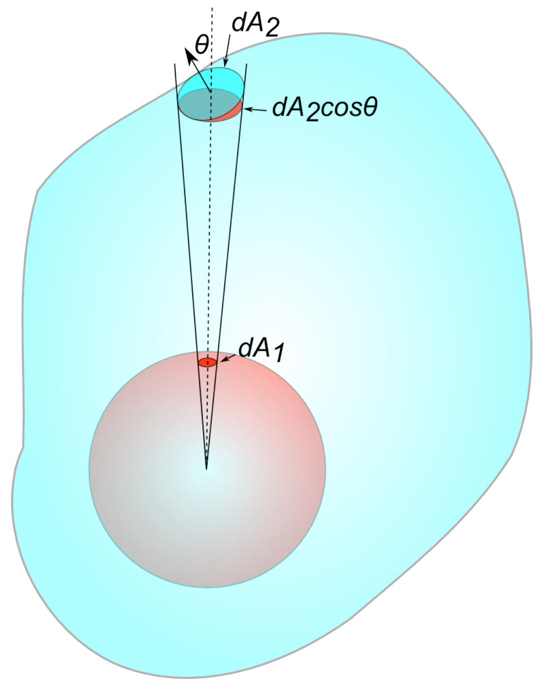
Gauss's law

A sphere of radius R is placed around a point charge q that is at the center of the sphere. Calculate the flux through the sphere. In this case at every point, the magnitude of the electric field is the same and it is perpendicular to the surface. Therefore the flux through the sphere is equal to the magnitude of the field at the surface of the sphere times the surface area of the sphere:

$$\Phi_E = \frac{q}{4\pi\varepsilon_o R^2} 4\pi R^2 = \frac{q}{\varepsilon_o}$$

Note that the flux is independent of the radius of the sphere that encloses the charge.

Now, consider the case where an arbitrary closed surface surrounds a point charge, as is depicted in the figure. We need to calculate the flux through each element of surface dA_2. We will show that the flux through dA_2 is equal to the flux through the surface element dA_1, which is on a spherical surface centered on the charge. If the flux through dA_2 is equal to the flux through dA_1, then the flux through the arbitrary closed surface will be equal to the flux through the sphere.



We can determine the ratio of the areas of dA_1 to dA_2. First consider the projection of dA_2 onto a surface that lies perpendicular to the radial direction, $dA_\perp$.

$$dA_\perp = dA_2 \cos\theta$$

But the ratio of the areas of $dA_\perp$ to dA_1 is equal to the square of the ratio of their distances from the charge, or

$$dA_\perp = dA_2 \cos\theta = \frac{r_2^2}{r_1^2} dA_1,$$

where r_1 and r_2 are the distance of A_1 and A_2 from the charge.

We can now compute the flux through dA_2.

$$d\Phi_{E2} = \vec{E} \cdot \overrightarrow{dA_2} = \frac{q}{4\pi\varepsilon_o r_2^2} dA_2 \cos\theta = \frac{q}{4\pi\varepsilon_o r_1^2} dA_1 = d\Phi_{E1}$$

When we then integrate the flux over the irregular surface, we find it is equal to the integral of the flux over the sphere. Therefore, for *any* closed surface surrounding a point charge, the flux through the surface is given by:

$$\Phi_E = \frac{q}{\varepsilon_o}$$

Finally, consider the case of multiple point charges inside a closed surface. Since the electric field at any point is the superposition of the fields of each of the individual charges, it follows that the flux through the surface equals the sum of the fluxes from each individual charge. Therefore, the flux through any closed surface is given by:

$$\Phi_E = \oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_o}$$

where Q is the total net charge enclosed within the sphere. This is Gauss's law. Note that Gauss's law follows from Coulomb's law (and vice versa).

If a closed surface is in an electric field but enclosed no net charge, then the flux through the surface will be zero. Show why.

Applications of Gauss's law

Infinite line of charge

$$E = \frac{\lambda}{2\pi\varepsilon_o r}$$

Infinite plane of charge

$$E = \frac{\sigma}{2\varepsilon_o}$$

Inside/outside a spherical shell of charge

E=0 inside

Non-conducting solid sphere of charge

$$E = \frac{Q}{4\pi\varepsilon_o r^2} \text{ for } r \geq a$$

$$E = \frac{Qr}{4\pi\varepsilon_o a^3} \text{ for } r < a$$

Charges on conductors

In a charged solid conductor, all the charge resides on the outside.

E is normal to the surface of a charged conductor

In a hollow conductor, all the charge resides on the outside. No electric field inside cavity. Image first a solid conducting sphere. We know that there is no electric field inside the conductor and at any point inside the conductor the net charge is zero. All the charge resides on the outside of the conductor. If the conductor is placed in an electric field, the electrons inside the conductor move in such a way that the field inside the conductor due to the charge distribution inside the conductor exactly cancels the external electric field. And since there is no net charge at any point inside the conductor, nothing inside the conductor contributes to the electric field. Therefore, you could remove any sections of the conductor, on the inside, without affecting the electric field in any way and at any point. Well, before we removed this material, there was no electric field inside the conductor, so, therefore, there will be no electric field inside the cavity after removing some of the material. This sometimes called a *Faraday cage*. We will see another proof of a Faraday cage after we study electric potential.

If we place a charge in the cavity of a hollow conductor, then the charge on the conductor will separate onto the two surfaces.